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A GEOMETRIC MODEL OF THE NONLINEAR EQUILIBRATION OF TWO - DIMENSIONAL EADY WAVES

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A Geometric Model of the Nonlinear Equilibration of Two-Dimensional Eady Waves.

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Abstract

The initial value problem for Eady's model is reexamined using a two-dimensional (x,z) Lagrangian semi-geostrophic model. A basic flow state in a periodic domain with a perturbation field imposed is represented by piecewise constant data. The Lagrangian conservation laws governing the motion allow the construction of a solution that describes the evolution of the unstable wave as it passes through the point of frontal collapse, reaches a maximum amplitude and then decays. The integration is continued and a second cycle is observed. A similar experiment was performed by Nakamura and Held, (1989), using a primitive equation model with diffusion. We carry out two integrations to investigate the sensitivity of our solution to the representation of the data by piecewise constant values and present observations and comparisons between our results and those of Nakamura and Held.

1. Introduction.

Nakamura and Held, 1989 (hereafter NH), studied the life cycle of an unstable Eady wave using a two-dimensional (x,z) primitive equation model that was capable of simultaneously producing realistic frontogenesis and equilibration. The kinematics was Eulerian, with diffusion included to suppress grid point noise and allow the integration to be continued through process of frontal collapse. The theory proposed to explain the life cycle of the wave is complex and thus the reader is referred to NH for detail. However, the salient argument in the theory involved the creation of a

potential vorticity (PV) source at the upper and lower boundaries, (Bretherton 1966). Diffusive mixing is then responsible for the transportation of air with enhanced PV into the interior of the domain and consequently stablizing the flow and reversing the sign of baroclinic energy conversion. In this note we present results from a model of this system with Lagrangian kinematics and semi-geostrophic dynamics. The absence of the explicit representation of diffusion in the equations of motion and the geometric structure of the model constitutes a significantly different approach from the work of NH.

Recently Blumen (1990,a,b), has studied the semi-geostrophic Eady model and incorporated momentum diffusion in the frontal zone. It is interesting to note that his model does not replicate the oscillatory equilibration or the change in the vertical structure of the wave that is produced by NH and ourselves.

Unlike NH or Blumen we do not present detailed arguments to explain the dynamics of the system. Our purpose is simply to present a brief outline of the structure of the model and the results. For a more detailed introduction to the geometric model see, for example, Holt (1990) or Cullen (1983).

2. Solution method and results.

Consider a domain D (x,y,z), which is taken to represent a region in the northern hemisphere (y pointing polewards). Let A be a cross-section,

$$A = \{ (x,z): -L < x < L, \; 0 < z < H \}$$

Periodicity is imposed across the lateral boundaries $x = \pm L$,

$$\phi(x + 2kL,z) = \phi(x,z)$$

$$\theta(x + 2kL,z) = \theta(x,z)$$

$$q(x + 2kL,z) = q(x,z)$$

where $\phi(x,z)$ is the geopotential, $\theta(x,z)$ the potential temperature, $q(x,z)$ is the potential vorticity and k any integer. We take a halfwidth $L = 2000km$ and depth $H = 10km$.

The potential temperature is taken to vary linearly with latitude,

$$\frac{\partial \theta}{\partial y} = -\frac{f\theta U}{gH} \qquad (1)$$

$U/H = u_g/z$, u_g the x-component of geostrophic wind, f the coriolis parameter (value $10^{-4}s^{-1}$) and g the gravitational acceleration.

In A, the dimensional form of the semi-geostrophic equations consistent with (1) can be written,

$$\frac{D\theta}{Dt} = -(fx - M)\frac{\partial\theta}{\partial y} \qquad (2)$$

$$\frac{DM}{Dt} = (z - H/2)\frac{fU}{H} \qquad (3)$$

$D/Dt = \partial/\partial t + u\partial/\partial x + w\partial/\partial z$ and $M(x,z)$ is the absolute momentum $(v_g + fx)$, where v_g is the y-component of the geostrophic wind.

The initial distribution of potential temperature is of the form,

$$\theta(x,z) = Bz + C\sin(\pi(x/L + z/H)) \qquad (4)$$

B and C are constants. The equations (1), (2), (3) and (4) together with a basic state of uniform potential vorticity, constitute a representation of the classic initial value Eady problem. Under these conditions strong gradients and discontinuities can develop in the fluid, so the numerical model must be capable of representing frontogenesis.

A conventional Eulerian model of the semi-geostrophic equations would produce solutions up to the point of frontal collapse. The presence of a discontinuity would then invalidate the Eulerian form of the equations of motion. The presence of a front in the Lagrangian sense is simply a boundary between air masses of different potential temperatures and does not invalidate the conservation laws used in our model to construct the solutions.

The PV, $q(x,z) = \delta(M,\theta)/\delta(x,z)$, is conserved in a region excluding frontal discontinuities (see Cullen & Purser 1984, pp 1487-1488). Consequently, the process of transportation of air with enhanced PV from the upper and lower boundaries, as discussed by NH, is feasible in our model as fluid initially on the boundaries may propagate into the interior frontal regions (Cullen & Purser, 1984), and will not violate PV conservation.

The geometric method discretizes the fluid in A into 'parcels' of equal area. To the i^{th} parcel is associated a value of absolute momentum $M_i(x,z)$ and potential temperature $\theta_i(x,z)$. The duality in representing the kinematics with either physical space coordinates (x,z) or with absolute momentum coordinates (M,θ) is given by a Legendre transformation (Chynoweth and Sewell 1989). The model constructs solutions, as described by Cullen and Purser (1989), by defining the distribution of $\rho(M,\theta) = q^{-1}(x,z)$ at each time and solving the (nonlinear) Monge-Ampere equation for the streamfunction of the geopotential in (M,θ) space with Neumann boundary conditions. The function ρ is updated using an advection equation,

$$\frac{\delta\rho}{\delta t} = -\mathbf{u}_g \cdot \nabla\rho$$

where $\nabla \Leftrightarrow (M,\theta)$. The physical solution is obtained by mapping the solution in (M,θ) space to (x,z) space. The only errors in the solution come from the piecewise constant representation of the data. The right hand sides of (2) and (3) are not constant over an element and have to be approximated by centroid values.

The model was run at two resolutions. The first discretized the flow into 240 elements (20x12 in x,z) and the second into 273 elements (21x13 in x,z). Figure 1 shows the evolution of PV through two cycles with the model running at the higher resolution. Figure 2 shows the corresponding arrangements of the parcels. Note the development of large positive PV near the surface once the front has formed. As the evolution progresses the tilt of the wave diminishes and eventually reverses sign. This phenomena was observed and discussed by NH.

Figure 3 shows the development of the maximum geostrophic flow during the integration. 3a and 3b give the results for the two resolutions. In both cases the first cycle is smooth and reminiscent of a growing normal mode instability, whereas one observes the second cycle as more variable. This is to be expected since the first cycle starts from normal mode initial conditions, while the second is triggered by the interaction of pre-existing disturbances at the upper and lower boundaries (see figure 1 at t = 341hrs). The detailed evolution of such disturbances is very dependent on the initial perturbations, which are non-modal (Farrell, 1984), and give rise to the vast range of observed cyclogenesis. We interpret one of the integrations as a control run and the other as a perturbation to the system generated by a small change in the initial data. Since the only errors in the solution come from the piecewise constant representation of the data, this experiment indicates the degree of uncertainty in the result. From this point of view the model seems to be very stable to perturbation. This result seems worthy of note in view of the sensitivity of the NH model to changes in the diffusion coefficients.

3. Summary.

We have outlined the results of modelling the Eady problem using a discrete, Lagrangian, semi-geostrophic model. With the absence of diffusion we can obtain an exact solution from piecewise constant data. The dynamics of the system is similar to that of NH and we obtain correspondingly similar results for the evolution of the system. Although we have approximated the flow in a piecewise constant manner the model produces a second cycle, the details of which appear to be very insensitive to perturbation in the initial data.

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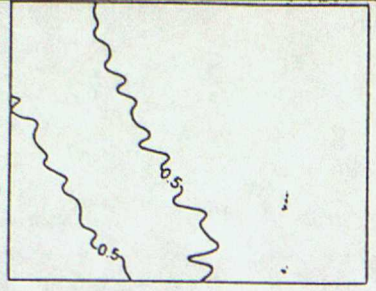
J. Atmos. Sci. **46**, 3055-3064.

FIGURE CAPTIONS.

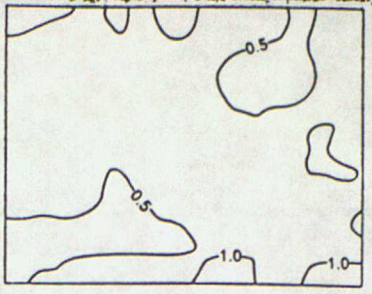
Figure 1. Evolution of potential vorticity (PV). Contour interval is in PV units (10^{-6} m^2 s^{-1} K Kg^{-1}). Time is given in hours.

Figure 2. Element configurations (with the elements initially on the upper and lower boundaries hatched). a) 0 hours, b) 205 hours, c) 341 hours, d) 597 hours.

Figure 3. Evolution of the maximum geostrophic velocity. a) At a resolution with 273 elements, and b) at a resolution with 240 elements.



0 H



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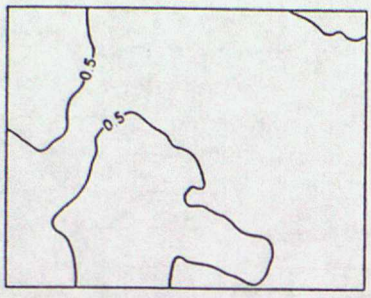

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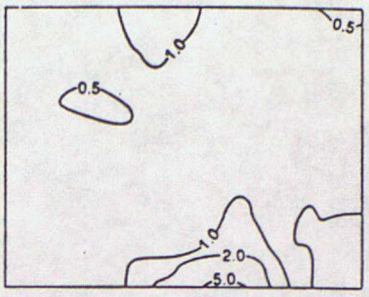

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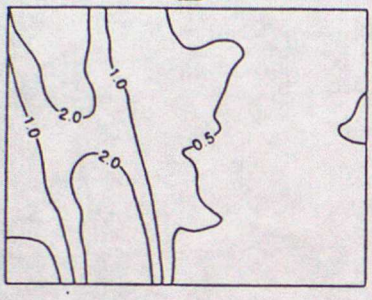

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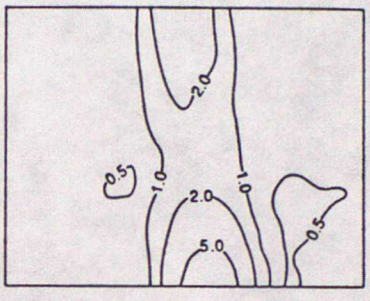

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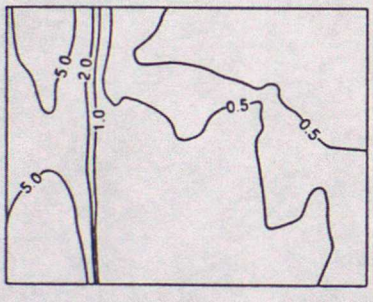

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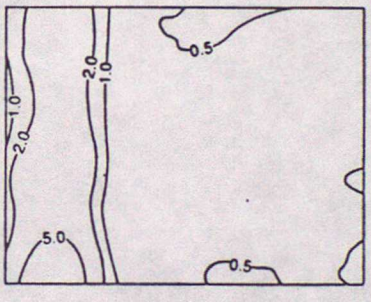

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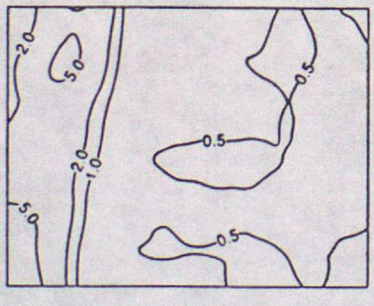

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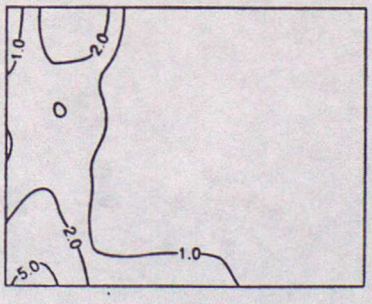

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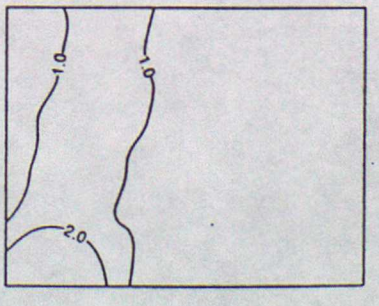
290 H



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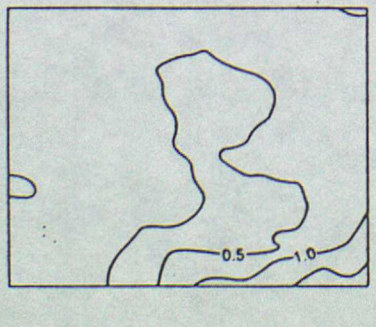
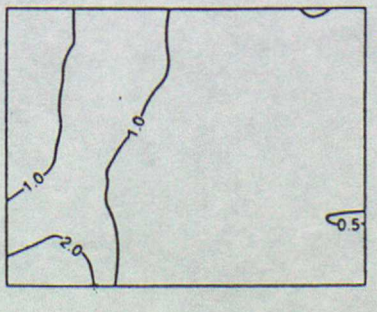
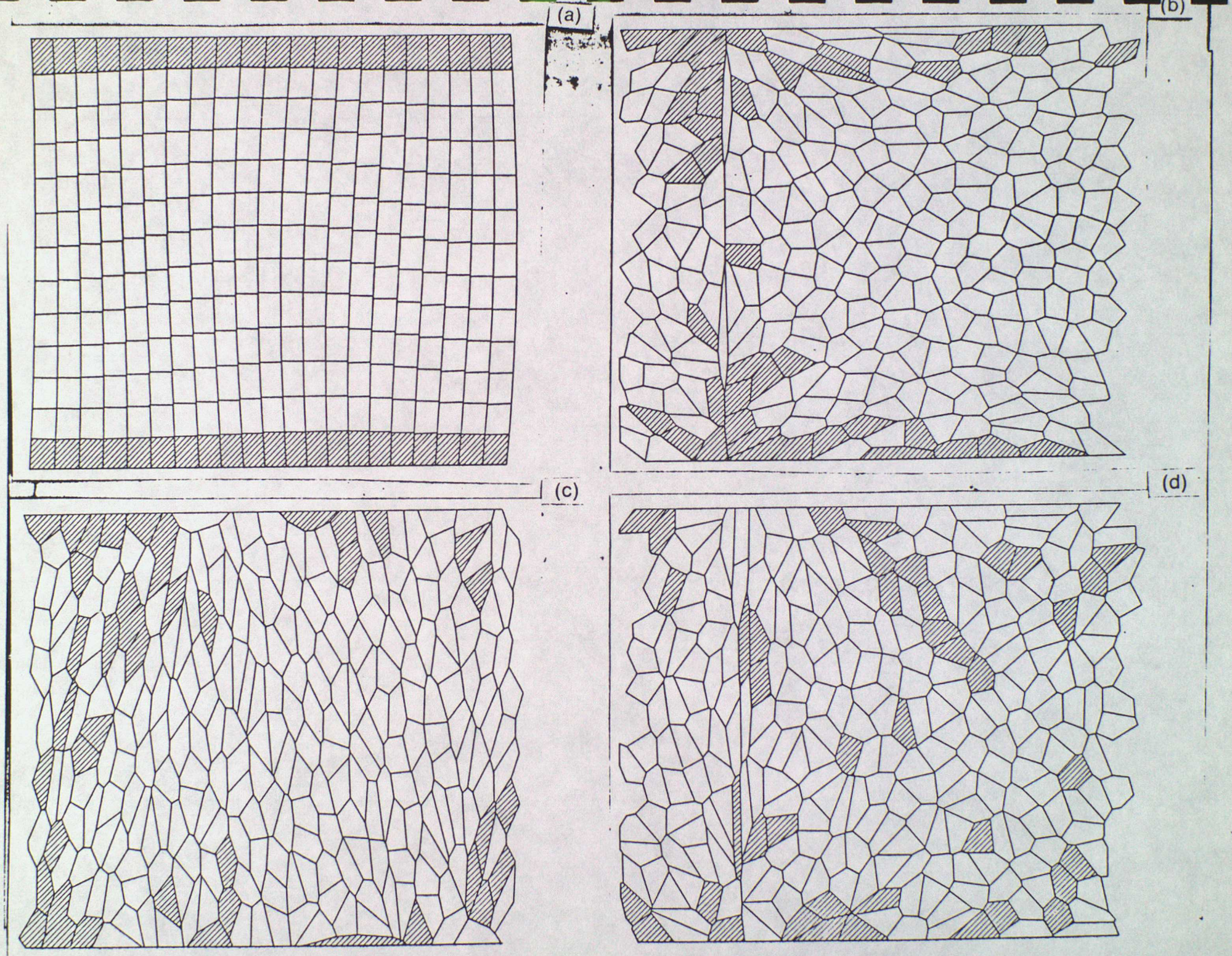


341 H

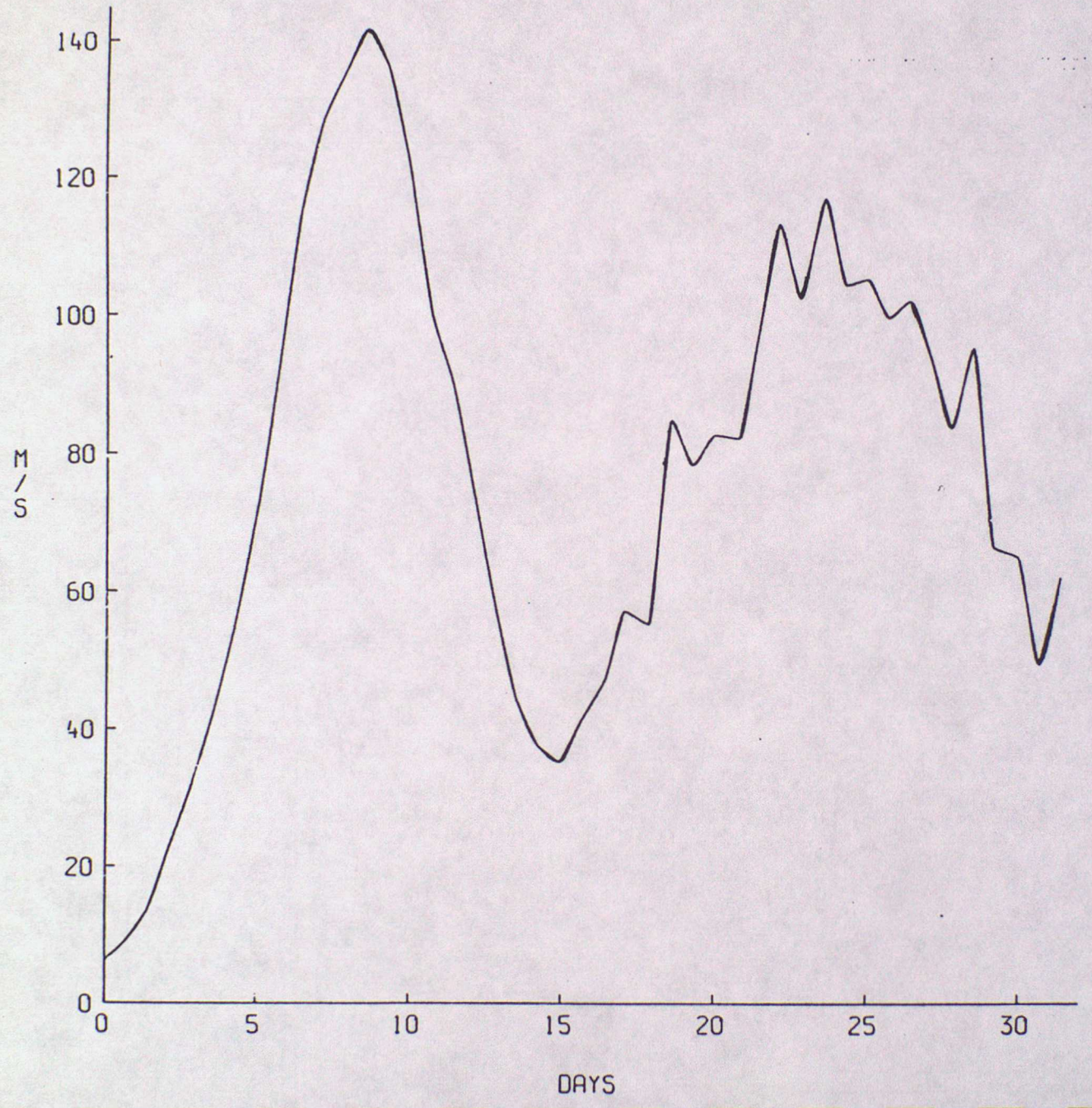


700 H

2



3 a)



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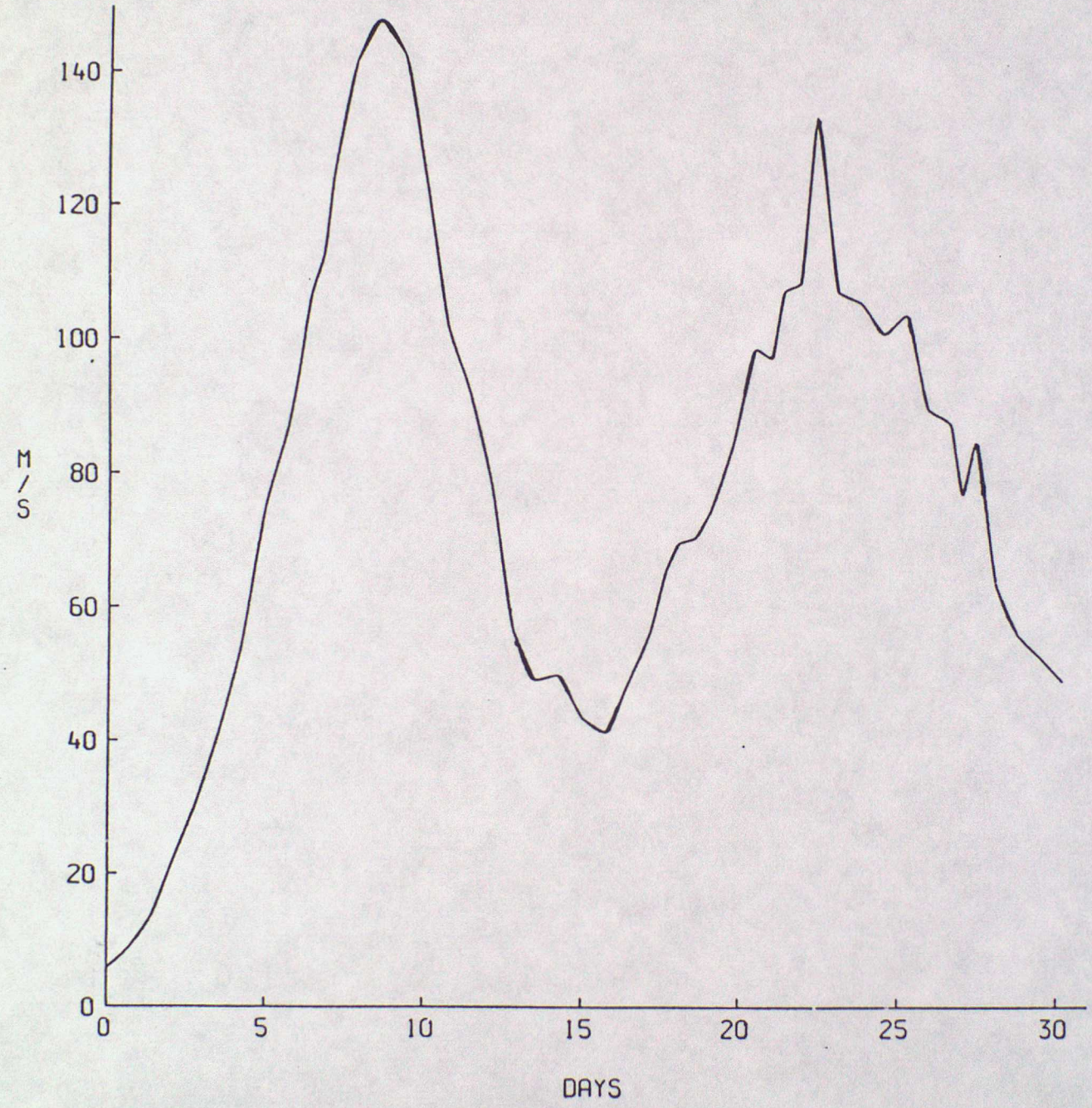

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October 1987

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M.J.P. Cullen
December 1987

3. **A geometric model of balanced, axisymmetric flows with embedded penetrative convection.**
G.J. Shutts, M. Booth and J. Norbury
February 1988

4. **Implicit finite difference methods for computing discontinuous atmospheric flows.**
M.J.P. Cullen
March 1988

5. **Variational aspects of semi-geostrophic theory.**
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6. **On the incorporation of atmospheric boundary layer effects into a balanced model.**
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8. **An analytical model of the growth of a frontal intrusion.**
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19. **Time - Integration Schemes for meteorological simulations involving oscillatory and dissipative modes.**
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20. **Mesoscale Model Experimental forecasts of the Haar of North East Scotland.**
S.P. Ballard, B.W.Golding, R.N.B. Smith
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21. **A conservative split-explicit integration scheme with fourth order horizontal advection.**
M.J.P. Cullen and T. Davies
October 1990

22. **A Geometric Model of the Nonlinear Equilibration of Two-Dimensional Eady Waves.**
M.J.P. Cullen and I. Roulstone
February 1991